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Local first integrals for systems of differential equations

Xiang Zhang

Department of Mathematics, Shanghai Jiaotong University, Shanghai 200030, People's Republic of China

E-mail: xzhang@mail.sjtu.edu.cn

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Abstract
The main purpose of this paper is to provide some sufficient conditions for a system of differential equations to have local first integrals in a certain neighbourhood of a singularity. Our results generalize those given in Kwek *et al* (2003 *Z. Angew. Math. Phys.* **54** 26) and Li *et al* (2003 *Z. Angew. Math. Phys.* **54** 235).

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1. Introduction and statement of the main results

Investigation of first integrals for systems of differential equations is a classical and vigorous work in the fields of both mathematics and physics. There is much research related to the integrability, partial integrability and nonintegrablity of differential systems (see, for instance, [3, 4, 6–19]). In those papers, combining the algebraic geometry, algebraic topology, differentiable manifold and singular analysis the authors developed many methods, such as Painlevé analysis, Colemmen embedding, Ziglin theory and so on, to solve some kinds of integrability problems.

We consider the following autonomous differential systems:

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) \qquad \mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{C}^n \tag{1}$$

where $\mathbf{F}$ is a vector-valued analytic function of dimension n satisfying $\mathbf{F}(\mathbf{0}) = \mathbf{0}$, i.e. $\mathbf{0}$ is a singularity of system (1), the dot denotes the derivative of $\mathbf{x}$ with respect to the time variable t. As usual, $\mathbb{C}$ is the field of complex numbers. If $\mathbf{F}(\mathbf{x})$ is a vector-valued formal series, system (1) is called a *formal system*. Generally, system (1) can be written as

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{f}(\mathbf{x}) \qquad \mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{C}^n \tag{2}$$

where $\mathbf{A}$ is the Jacobian matrix $D\mathbf{F}(\mathbf{0})$ of the vector field $\mathbf{F}(\mathbf{x})$ at $\mathbf{x} = \mathbf{0}$ and $\mathbf{f}(\mathbf{x}) = O(\|\mathbf{x}\|^2)$.

Poincaré [16] in 1891 obtained that if $\mathbf{A}$ is diagonal and its eigenvalues do not satisfy any resonant conditions, then system (2) does not have analytic first integrals in a neighbourhood of $\mathbf{0}$. In 1996 Furta [8] provided an elementary proof of this result. Recently, Kwek *et al* [13] generalized this result to the case that system (2) has m $(m < n)$ functionally independent analytic first integrals, and that $\mathbf{A}$ is diagonalizable and its eigenvalues satisfy exactly m linearly independent resonant relations. Li *et al* [14] extended the Poincaré result to the case that an eigenvalue of the matrix $\mathbf{A}$ is zero and the other eigenvalues are non-resonant. In this paper we will generalize these results.

Let $U \subset \mathbb{C}^n$ be an open connected subset. A non-constant analytic function $H : U \to \mathbb{C}$ is called an *analytic first integral* of system (1) in U if and only if along every solution curve

$$\left\langle \frac{\partial H}{\partial \mathbf{x}}, \mathbf{F}(\mathbf{x}) \right\rangle \equiv 0 \qquad \text{i.e.} \quad \sum_{i=1}^{n} F_i(\mathbf{x}) \frac{\partial H}{\partial x_i} \equiv 0 \quad \text{in } U \tag{3}$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product of vectors in $\mathbb{C}^n$. If $H(\mathbf{x})$ is a formal series in $\mathbf{x}$ and satisfies (3), then $H(\mathbf{x})$ is called a *formal first integral* of system (1) in a neighbourhood of the singularity $\mathbf{x} = \mathbf{0}$. Obviously, if a formal first integral is convergent, it is an analytic first integral. In what follows, we require, without loss of generality, that all mentioned first integrals do not have constant terms. We say that first integrals of system (1) in U are *independent* if the rank of their Jacobian matrix in U is equal to the number of the first integrals.

Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of the matrix $\mathbf{A}$. We say that the n eigenvalues satisfy a *resonant condition* if there exist $s \in \{1, \ldots, n\}$ and $k_1, \ldots, k_n \in \mathbb{Z}^+ = \mathbb{N} \cup \{0\}$ with $\sum_{i=1}^n k_i > 1$ such that $\lambda_s = \sum_{i=1}^n k_i \lambda_i$, where $\mathbb{N}$ denotes the set of natural numbers. Set

$$\mathcal{G} = \left\{ (k_1, \ldots, k_n); \sum_{i=1}^{n} k_i \lambda_i = 0, k_i \in \mathbb{Z}^+, i = 1, \ldots, n \right\}$$

$$\mathcal{G}' = \left\{ (k_2, \ldots, k_n); \sum_{i=2}^{n} k_i \lambda_i = 0, k_i \in \mathbb{Z}^+, i = 2, \ldots, n \right\}.$$

We say that $\lambda_1, \ldots, \lambda_n$ satisfy a *resonant relation* if $\sum_{i=1}^n k_i \lambda_i = 0$ with $k_i \in \mathbb{Z}^+$ and $\sum_{i=1}^n k_i \geqslant 1$.

Our first result is the following:

Theorem 1. *Assume that system (2) is analytic and has m $(m < n)$ nontrivial locally analytic first integrals $\Phi_1(\mathbf{x}), \ldots, \Phi_m(\mathbf{x})$ in a neighbourhood of the singularity $\mathbf{x} = \mathbf{0}$. If the m first integrals are independent and the linear space generated from $\mathcal{G}$ has dimension m, then any nontrivial analytic first integral of system (2) is an analytic function in $\Phi_1(\mathbf{x}), \ldots, \Phi_m(\mathbf{x})$.*

We note that if $m = n - 1$, system (2) is completely integrable in a neighbourhood of $\mathbf{x} = \mathbf{0}$. Consequently, all the solution curves are given by $\{\Phi_1(\mathbf{x}) = c_1\} \cap \cdots \cap \{\Phi_m(\mathbf{x}) = c_m\}$, where $c_1, \ldots, c_m$ are suitable constants.

We remark that this result generalizes theorem A of [13] because we do not need the condition that $\mathbf{A}$ is diagonalizable, which is required in theorem A of [13]. In addition, our proof is much simpler than that given in [13]. We note that theorem 1 is also correct if we change analytic in theorem 1 into formal.

The following simple example provides an application of theorem 1:

Example 1. Consider the following system:

$$\dot{x} = 0 \qquad \dot{y} = 0 \qquad \dot{z} = ax + by + cz + f(x, y, z) \tag{4}$$

where f is an analytic function and $f = O(\|(x, y, z)\|^2)$. Obviously, $H_1 = x$ and $H_2 = y$ are two analytic first integrals of the last system, and they are independent. We know that the eigenvalues of the linear part of the system are $\lambda_1 = 0$, $\lambda_2 = 0$ and $\lambda_3 = c$. If $c \neq 0$, the space of solutions (k_1, k_2, k_3) of the linear equation $k_1\lambda_1 + k_2\lambda_2 + k_3\lambda_3 = 0$ has dimension 2. It is easy to check that any analytic first integral of (4) is an analytic function in H_1 and H_2, i.e. x and y.

This example shows that the conditions of theorem 1 are sufficient, but not necessary.

Our next result is an extension of theorem 1 in [14].

Theorem 2. *Assume that system (2) is analytic and has m* $(m < n-1)$ *nontrivial independent analytic first integrals* $\Phi_1(\mathbf{x}), \ldots, \Phi_m(\mathbf{x})$ *in a neighbourhood of the singularity* $\mathbf{x} = \mathbf{0}$*, and that* λ_1 *is a zero eigenvalue and the linear space generated by* $\mathcal{G}'$ *has dimension m. If the eigenspace associated with* λ_1 *is tangent to the* $(n-m)$*-dimensional surface formed by* $S = \{\Phi_1(\mathbf{x}) = 0\} \cap \cdots \cap \{\Phi_m(\mathbf{x}) = 0\}$*, then the following statements hold:*

(a) For $m < n-2$*, system (2) has a formal first integral in a neighbourhood of* $\mathbf{x} = \mathbf{0}$ *which is a formal series of the form*

$$H(\mathbf{x}) = \sum_{|\mathbf{s}|=1}^{\infty} h_{\mathbf{s}}(\mathbf{x})\Phi_1^{s_1}(\mathbf{x})\ldots\Phi_m^{s_m}(\mathbf{x}) \tag{5}$$

with $h_{\mathbf{s}}(\mathbf{x})$ *not all constants for* $\mathbf{s} = (s_1, \ldots, s_m) \in (\mathbb{N}^+)^m$ *and* $|\mathbf{s}| = s_1 + \cdots + s_m$*, if and only if the singularities of system (2) form a surface which is transverse to S and passes through the origin* $\mathbf{x} = \mathbf{0}$.

(b) For $m = n-2$*, system (2) has an analytic first integral in a neighbourhood of* $\mathbf{x} = \mathbf{0}$ *which is an analytic function of form (5) with* $h_{\mathbf{s}}(\mathbf{x})$ *not all constants for* $\mathbf{s} \in (\mathbb{N}^+)^{\mathbf{m}}$*, if and only if the singularities of system (2) form a surface which is transverse to S and passes through the origin* $\mathbf{x} = \mathbf{0}$.

The following example provides an application of the last theorem:

Example 2. Consider the following analytic system:

$$\dot{x} = yz - xz + x^3 - x^2y \qquad \dot{y} = 0 \qquad \dot{z} = z - x^2. \tag{6}$$

The eigenvalues of the system at the origin are $\lambda_1 = 0$, $\lambda_2 = 0$ and $\lambda_3 = 1$. The space of solutions (k_2, k_3) of the linear equation $k_2\lambda_2 + k_3\lambda_3 = 0$ has dimension 1. The eigenspace corresponding to λ_1 is tangent to the plane $y = 0$. The parabolic cylinder $z = x^2$ is full of the singularities of system (6), and it is transverse to the invariant plane $y = 0$. Consequently, it follows from theorem 2 that the system has an analytic first integral in a neighbourhood of the singularity $\mathbf{0}$. In fact, we also can get the result from the fact that system (6) has the same first integral as that of the following system:

$$\dot{x} = y - x \qquad \dot{y} = 0 \qquad \dot{z} = 1.$$

The latter is regular at the origin.

Our following result is an extension of theorem B in [13]. System (1) is said to be *quasi-homogeneous* of degree l with exponents $s_1, \ldots, s_n \in \mathbb{Z}\backslash\{0\}$ and $l \in \mathbb{N}\backslash\{1\}$, if for any $\rho \in \mathbb{R}^+$ and $\mathbf{x} \in \mathbb{C}^n$ we have that $\rho^{\mathbf{E}-\mathbf{S}}\mathbf{F}(\mathbf{x}) = (\rho^{1-s_1}F_1, \ldots, \rho^{1-s_n}F_n)$ is quasi-homogeneous of degree l, i.e.

$$F_i(\rho^{s_1}x_1, \rho^{s_2}x_2, \ldots, \rho^{s_n}x_n) = \rho^{s_i+l-1}F_i(x_1, x_2, \ldots, x_n) \qquad i = 1, 2, \ldots, n \tag{7}$$

where $\rho^{\mathbf{E}-\mathbf{S}} = \mathrm{diag}(\rho^{1-s_1}, \ldots, \rho^{1-s_n})$, $\mathbf{E}$ the unit matrix and $\mathbf{S} = \mathrm{diag}(s_1, \ldots, s_n)$. We call $s_i + l - 1$ the *weight degree* of F_i, and $(s_1, \ldots, s_n)$ the *weight exponents*. System (1) is called *semi-quasi-homogeneous* of degree l with the weight exponents $(s_1, \ldots, s_n)$ if

$$\mathbf{F}(\mathbf{x}) = \mathbf{F}_l(\mathbf{x}) + \widetilde{\mathbf{F}}(\mathbf{x}) \tag{8}$$

with $\rho^{\mathbf{E}-\mathbf{S}}\mathbf{F}_l$ the vector-valued quasi-homogeneous polynomial of degree l and $\rho^{\mathbf{E}-\mathbf{S}}\widetilde{\mathbf{F}}(\mathbf{x})$ the sum of vector-valued quasi-homogeneous polynomials of degree all larger than l or all less than l. In the former (respectively, latter), system (1) is called *positively* (respectively, *negatively*) *semi-quasi-homogeneous*. $\mathbf{F}_l(\mathbf{x})$ is called the *first quasi-homogeneous term* of $\mathbf{F}(\mathbf{x})$ associated with the weight exponents $(s_1, \ldots, s_n)$. Let H be an analytic or a formal first integral of system (1) with $\mathbf{F}$ being of type (8) associated with the weight exponents $(s_1, \ldots, s_n)$. According to this weight exponent we can rewrite H as $H = H_m + \widetilde{H}$, where H_m is the quasi-homogeneous component of H with weight degree m, and $\widetilde{H}$ is the summation of quasi-homogeneous polynomials of weight degree all larger than m or all less than m depending on system (1) positively or negatively. Then the first quasi-homogeneous term H_m of H is a first integral of $\dot{\mathbf{x}} = \mathbf{F}_l(\mathbf{x})$. Every non-zero solution $\mathbf{c}$ of the algebraic system

$$\mathbf{F}_l(\mathbf{c}) + \mathbf{W}\mathbf{c} = \mathbf{0} \tag{9}$$

is called a *balance* associated with system (1) with $\mathbf{F}$ being of form (8), where $\mathbf{W} = \mathbf{S}/(l-1)$. The matrix $\mathbf{K} = D\mathbf{F}_l(\mathbf{c}) + \mathbf{W}$ is the so-called *Kowalevskaya matrix*, where $D\mathbf{F}_l(\mathbf{c})$ is the Jacobian matrix of $\mathbf{F}_l$ at $\mathbf{x} = \mathbf{c}$. We call the eigenvalues of $\mathbf{K}$ the *Kowalevskaya exponents*.

Theorem 3. *Assume that system (1) is semi-quasi-homogeneous of degree l associated with the weight exponents $(s_1, \ldots, s_n)$ having $\mathbf{F}$ being of type (8), and that it has m ($m < n-1$) nontrivial analytic first integrals $\Phi_1(\mathbf{x}), \ldots, \Phi_m(\mathbf{x})$ in a neighbourhood of the singularity $\mathbf{x} = \mathbf{0}$. Denote by $\Phi_1^l(\mathbf{x}), \ldots, \Phi_m^l(\mathbf{x})$ the first quasi-homogeneous terms of $\Phi_1(\mathbf{x}), \ldots, \Phi_m(\mathbf{x})$ respectively. Moreover, we suppose that the following conditions hold:*

(i) There exists a balance $\bar{\mathbf{c}}$ such that the corresponding Kowalevskaya exponents $\lambda_1, \ldots, \lambda_n$ satisfy the conditions: $\lambda_1 = 0$, and the set $\mathcal{G}'$ of the vectors $(k_2, \ldots, k_n)$ satisfying the conditions

$$\sum_{i=2}^{n} k_i\lambda_i = 0 \qquad \sum_{i=2}^{n} k_i \neq 0 \qquad \text{and} \qquad k_i \in \mathbb{Z}^+ \quad i = 2, \ldots, n$$

has the rank m.

(ii) The eigenspace corresponding to λ_1 is tangent to the manifold given by $S = \{\Phi_1^l(\mathbf{x}) = 0\} \cap \ldots \cap \{\Phi_m^l(\mathbf{x}) = 0\}$.

(iii) $\Phi_1^l(\mathbf{x}), \ldots, \Phi_m^l(\mathbf{x})$ are independent at the balance $\bar{\mathbf{c}}$.

Then if the balance $\bar{\mathbf{c}}$ is an isolated solution of (9), any first integral (analytic or formal series) of system (1) in a neighbourhood of the singularity $\mathbf{0}$ is an analytic function or a formal series in $\Phi_1(\mathbf{x}), \ldots, \Phi_m(\mathbf{x})$.

This paper is organized as follows. In section 2 we recall some elementary tools for proving our results. In sections 3 and 4 we prove theorems 1 and 2, respectively. The proof of theorem 3 is given in section 5. A conclusion is stated in the last section

2. Elementary tools

In the proof of our main results, we need the following lemma (for a proof, see [14]):

Lemma 4. *Let* $\mathbf{A}$ *be an* $n \times n$ *matrix, and let* $\lambda_1, \lambda_2, \ldots, \lambda_n$ *be its eigenvalues. Let* Θ_k *be the linear space formed by the homogeneous polynomials of degree k with* $k \geqslant 1$ *in* $\mathbb{C}[x_1, \ldots, x_n]$*, the ring of complex polynomials in the variables* $\mathbf{x} = (x_1, x_2, \ldots, x_n)$*. We define a linear operator from* Θ_k *into itself given by*

$$L(h)(\mathbf{x}) = \left\langle \frac{\partial h}{\partial \mathbf{x}}, \mathbf{A}\mathbf{x} \right\rangle$$

for $h \in \Theta_k$*. Then the set of eigenvalues of L is* $\Omega = \left\{\sum_{i=1}^n k_i \lambda_i, k_i \in \mathbb{Z}^+, \sum_{i=1}^n k_i = k\right\}$.

The following classical theorem will be used in the proof of theorem 2 (see for instance, [1, 2, 5]):

Poincaré–Dulac theorem. *If* $\mathbf{F}(\mathbf{x}) = \mathbf{A}\mathbf{x} + \cdots$ *is a formal series and* $\mathbf{A}$ *is a Jordan normal matrix, then system (1) can be reduced to the canonical form*

$$\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} + \mathbf{w}(\mathbf{y}) \tag{10}$$

by means of a formal change of variables $\mathbf{x} = \mathbf{y} + \cdots$*, where* $\mathbf{w}(\mathbf{y}) = (w_1(\mathbf{y}), \ldots, w_n(\mathbf{y}))$ *and all monomials* $\mathbf{y}^m = y_1^{m_1}, \ldots, y_n^{m_n}$ *in the series* $w_i(\mathbf{y})$ *for all i are resonant in the sense that* $\lambda_i = \langle m, \lambda \rangle$ *with* $|m| \geqslant 2$*, where* $\lambda = (\lambda_1, \ldots, \lambda_n)$ *are eigenvalues of* $\mathbf{A}$, $m = (m_1, \ldots, m_n)$ *and* $|m| = \sum_{i=1}^n m_i$.

3. Proof of theorem 1

Since $\Phi_1(\mathbf{x}), \ldots, \Phi_n(\mathbf{x})$ are independent, we can take the change of variables

$$\overline{y}_i = \Phi_i(x_1, \ldots, x_n) \qquad i = 1, \ldots, m$$

$$\begin{pmatrix} \overline{y}_{m+1} \\ \vdots \\ \overline{y}_n \end{pmatrix} = \mathbf{M}\mathbf{x}$$

with $\mathbf{M}$ an $(n-m) \times n$ matrix such that the Jacobian matrix of the transformation is non-zero at $\mathbf{x} = \mathbf{0}$. Then under this transformation, system (2) becomes

$$\dot{\overline{\mathbf{y}}}_m = \mathbf{0} \qquad \dot{\overline{\mathbf{y}}}_{n-m} = \overline{\mathbf{B}}\overline{\mathbf{y}} + \overline{\mathbf{g}}(\overline{\mathbf{y}}) \tag{11}$$

where $\overline{\mathbf{y}}_m = (\overline{y}_1, \ldots, \overline{y}_m)$, $\overline{\mathbf{y}}_{n-m} = (\overline{y}_{m+1}, \ldots, \overline{y}_n)$, $\overline{\mathbf{y}} = (\overline{\mathbf{y}}_m, \overline{\mathbf{y}}_{n-m})$, $\overline{\mathbf{B}}$ is an $(n-m) \times n$ matrix, and $\overline{\mathbf{g}}(\overline{\mathbf{y}})$ an vector-valued analytic function in $\overline{\mathbf{y}}$ with $\overline{\mathbf{g}}(\overline{\mathbf{y}}) = O(\|\overline{\mathbf{y}}\|^2)$. From the assumption of the theorem, it follows that the matrix $\overline{\mathbf{B}}$ has an $n-m$ square submatrix such that its eigenvalues do not satisfy any resonant relations. Hence, we can take an invertible linear transformation such that (11) has the following form:

$$\dot{\mathbf{y}}_m = 0 \qquad \dot{\mathbf{y}}_{n-m} = \mathbf{B}\mathbf{y}_{n-m} + \mathbf{g}(\mathbf{y}) \tag{12}$$

where $\mathbf{y}_m = (y_1, \ldots, y_m)$, $\mathbf{y}_{n-m} = (y_{m+1}, \ldots, y_n)$, $\mathbf{B}$ is an $n-m$ square matrix and $\mathbf{g}(\mathbf{y}) = \mathbf{O}(\|\mathbf{y}\|^2)$. Now the $n-m$ eigenvalues of $\mathbf{B}$ do not satisfy any resonant relations.

From the construction of system (12) we know that system (2) has an analytic first integral in $\Phi_1, \ldots, \Phi_m$ if and only if system (12) has an analytic first integral in $y_1, \ldots, y_m$. So in what follows, we will give the proof for system (12).

Assume that $H(\mathbf{y})$ is an analytic first integral of system (12) in a neighbourhood of $\mathbf{y} = \mathbf{0}$. Without loss of generality we can write $H(\mathbf{y})$ in the following form:

$$H(\mathbf{y}) = \sum_{|\mathbf{s}|=0}^{\infty} a_{\mathbf{s}}(\mathbf{y}_{n-m})\mathbf{y}_m^{\mathbf{s}} \tag{13}$$

where $a_{\mathbf{s}}(\mathbf{y}_{n-m})$ are analytic functions in $\mathbf{y}_{n-m}, \mathbf{y}_m^{\mathbf{s}} = y_1^{s_1} \dots y_m^{s_m}, s_i \in \mathbb{Z}^+$ and $|\mathbf{s}| = s_1 + \cdots + s_m$. From the definition of first integrals we get that

$$\left\langle \frac{\partial H(\mathbf{y})}{\partial \mathbf{y}_{n-m}}, \mathbf{B}\mathbf{y}_{n-m} + \mathbf{g}(\mathbf{y}) \right\rangle \equiv 0.$$

Comparing the terms with $\mathbf{y}_m^{\mathbf{0}}$ gives

$$\left\langle \frac{\partial a_{\mathbf{0}}(\mathbf{y}_{n-m})}{\partial \mathbf{y}_{n-m}}, \mathbf{B}\mathbf{y}_{n-m} \right\rangle \equiv 0.$$

Generally, we can set $a_{\mathbf{0}}(\mathbf{y}_{n-m}) = c_0 + c_k(\mathbf{y}_{n-m}) + o(\mathbf{y}_{n-m})$, where $c_k(\mathbf{y}_{n-m})$ is a homogeneous polynomial of degree k in $\mathbf{y}_{n-m}$, and $o(\mathbf{y}_{n-m})$ denotes the summation of terms in $\mathbf{y}_{n-m}$ with degree larger than k. Hence, we have

$$L_2[c_k](\mathbf{y}_{n-m}) = \left\langle \frac{\partial c_k(\mathbf{y}_{n-m})}{\partial \mathbf{y}_{n-m}}, \mathbf{B}\mathbf{y}_{n-m} \right\rangle \equiv 0.$$

Since the eigenvalues of $\mathbf{B}$ do not satisfy any resonant relations, it follows from lemma 4 that the linear operator L_2 is invertible. Therefore, we have $c_k(\mathbf{y}_{n-m}) = 0$. This means that $a_0(\mathbf{y}_{n-m})$ is a constant.

By induction we can prove that all the $a_{\mathbf{s}}(\mathbf{y}_{n-m})$ in (13) are constants. So, $H(\mathbf{y})$ depends on $\mathbf{y}_m$ only. Consequently, if system (2) has an analytic first integral, it must be an analytic function in $\Phi_1, \dots, \Phi_m$. This proves the theorem.

4. Proof of theorem 2

We first prove statement (a). Since λ_1 is a zero eigenvalue, there exists an invertible matrix $\mathbf{M}$ such that

$$\mathbf{A}^* = \mathbf{M}^{-1}\mathbf{A}\mathbf{M} = \begin{pmatrix} 0 & \mathbf{0}_{n-1} \\ \mathbf{0}_{n-1}^{\mathrm{T}} & \overline{\mathbf{B}} \end{pmatrix}$$

is a Jordan normal form of the matrix $\mathbf{A}$, where $\mathbf{0}_{n-1}$ is a line zero vector of dimension $n-1$, T denotes the transpose of a matrix, and $\overline{\mathbf{B}}$ is an $n-1$ square matrix. Applying the change of variables $\mathbf{y} = \mathbf{M}^{-1}\mathbf{x}$ to system (2) we get the following equivalent system:

$$\dot{\mathbf{y}} = \mathbf{A}^*\mathbf{y} + \mathbf{q}(\mathbf{y}). \tag{14}$$

Consequently, system (14) has the first integrals $\Phi_1^*(\mathbf{y}) = \Phi_1(\mathbf{M}\mathbf{y}), \dots, \Phi_m^*(\mathbf{y}) = \Phi_m(\mathbf{M}\mathbf{y})$.

We make the following analytic change of variables:

$$\begin{pmatrix} \overline{y}_1 \\ \vdots \\ \overline{y}_{n-m} \end{pmatrix} = \mathbf{M}'\mathbf{y} \qquad \overline{y}_i = \Phi_{i-n+m}^*(\mathbf{y}) \qquad i = n-m+1, \dots, n$$

with $\mathbf{M}'$ an $(n-m) \times n$ matrix, such that the Jacobian matrix of the transformation is invertible at $\mathbf{y} = \mathbf{0}$, and that system (14) has the following form:

$$\dot{\overline{\mathbf{y}}}_1 = \begin{pmatrix} 0 & \mathbf{0}_{n-1} \\ \mathbf{0}_{n-1}^{\mathrm{T}} & \mathbf{B} \end{pmatrix} \overline{\mathbf{y}}_1 + \overline{\mathbf{q}}(\overline{\mathbf{y}}) \qquad \dot{\overline{\mathbf{y}}}_2 = 0 \tag{15}$$

where $\overline{\mathbf{y}}_1 = (\overline{y}_1, \dots, \overline{y}_{n-m}), \overline{\mathbf{y}}_2 = (\overline{y}_{n-m+1}, \dots, \overline{y}_n), \overline{\mathbf{y}} = (\overline{\mathbf{y}}_1, \overline{\mathbf{y}}_2)$, $\mathbf{B}$ is an $n-m-1$ square matrix and $\overline{\mathbf{q}}(\overline{\mathbf{y}})$ a vector-valued function of dimension $n-m$ with $\overline{\mathbf{q}}(\overline{\mathbf{y}}) = O(\|\overline{\mathbf{y}}\|^2)$. Let $\mu_2, \dots, \mu_{n-m}$ be the $n-m-1$ eigenvalues of the matrix $\mathbf{B}$. Since we assume that the $n-1$ eigenvalues $\lambda_2, \dots, \lambda_n$ of $\mathbf{A}$ satisfy exactly m resonant relations, it follows from the transformation, changing (2) into (15), that the $n - m - 1$ eigenvalues $\mu_2, \ldots, \mu_{n-m}$ do not satisfy any resonant relations.

We note that system (1) has a C^r first integral in a neighbourhood of the singularity $\mathbf{x} = \mathbf{0}$ if and only if system (15) has a C^r first integral in a neighbourhood of the singularity $\overline{\mathbf{y}} = \mathbf{0}$. So, in the following we prove our results for system (15).

The Poincaré–Dulac theorem implies that there exists a formal series $\overline{\mathbf{y}} = \mathbf{z} + \cdots$ such that system (15) can be reduced to the following canonical form:

$$\dot{z}_1 = h_1(z_1, \mathbf{z}_2, \mathbf{z}_3) \qquad \dot{\mathbf{z}}_2 = \mathbf{B}\mathbf{z}_2 + \mathbf{h}_2(z_1, \mathbf{z}_2, \mathbf{z}_3) \qquad \dot{\mathbf{z}}_3 = 0 \tag{16}$$

where $\mathbf{z} = (z_1, \mathbf{z}_2, \mathbf{z}_3), \mathbf{z}_2 = (z_2, \ldots, z_{n-m}), \mathbf{z}_3 = (z_{n-m+1}, \ldots, z_n) = \overline{\mathbf{y}}_2$ and $\mathbf{h}_2 = (h_2, \ldots, h_{n-m})$ are vector-valued functions of dimension $n - m - 1$. The series h_1 and $\mathbf{h}_2$ start with the terms of degree at least 2. All monomials in the series h_1 and $\mathbf{h}_2$ are resonant in z_1 and $\mathbf{z}_2$ with coefficients of functions in $\mathbf{z}_3$.

From the definition of resonant monomials $z_1^{k_1}, \ldots, z_{n-m}^{k_{n-m}}$ and the Poincaré–Dulac theorem, we obtain that $0 = \lambda_1 = \langle k, \lambda \rangle = \sum_{i=2}^{n-m} k_i \mu_i$, where $k = (k_1, \ldots, k_{n-m})$ and $\lambda = (\lambda_1, \mu_2, \ldots, \mu_{n-m})$. This condition is equivalent to $k_2 = \cdots = k_{n-m} = 0$, because $\mu_2, \ldots, \mu_{n-m}$ do not satisfy any resonant relations. So $h_1(z_1, \mathbf{z}_2, \mathbf{z}_3) = h_1(z_1, \mathbf{z}_3)$, in which the term of the lowest order has degree at least 2. Similarly, we can prove that $\mathbf{h}_2(z_1, \mathbf{0}_{n-m-1}, \mathbf{z}_3) \equiv 0$.

We first prove the 'only if' part. It is easy to prove that system (15) has a formal first integral in the neighbourhood of $\overline{\mathbf{y}} = \mathbf{0}$ if and only if system (16) has a formal first integral in a neighbourhood of $\mathbf{z} = \mathbf{0}$. By the assumption we can assume that system (16) has a formal first integral $H(z_1, \mathbf{z}_2, \mathbf{z}_3)$ and that H has the form

$$H(z_1, \mathbf{z}_2, \mathbf{z}_3) = H_1(z_1, \mathbf{z}_2) + H_2(z_1, \mathbf{z}_2, \mathbf{z}_3) \qquad H_1(z_1, \mathbf{z}_2) = \sum_{i=0}^{\infty} a_i(\mathbf{z}_2) z_1^i$$

where $a_i(\mathbf{z}_2)$ are formal series in $\mathbf{z}_2$ and $H_2(z_1, \mathbf{z}_2, \mathbf{0}_3) \equiv 0$. Let

$$\mathbf{h}_2(z_1, \mathbf{z}_2, \mathbf{z}_3) = \sum_{i=0}^{\infty} \mathbf{b}_i(\mathbf{z}_2, \mathbf{z}_3) z_1^i$$

where $\mathbf{b}_i(\mathbf{z}_2, \mathbf{z}_3)$ are vector-valued formal series in $\mathbf{z}_2$ of dimension $n - m - 1$ with coefficients of functions in $\mathbf{z}_3$, and $\mathbf{b}_0(\mathbf{z}_2, \mathbf{z}_3)$ has degree at least 2 in $\mathbf{z}_2$. From the definition of first integrals we obtain that

$$\left(\sum_{i=1}^{\infty} \mathrm{i} a_i(\mathbf{z}_2) z_1^{i-1} + \frac{\partial H_2}{\partial z_1} \right) h_1(z_1, \mathbf{z}_3) + \left\langle \sum_{i=0}^{\infty} \frac{\partial a_i(\mathbf{z}_2)}{\partial \mathbf{z}_2} z_1^i + \frac{\partial H_2}{\partial z_2}, \mathbf{B}\mathbf{z}_2 + \sum_{i=0}^{\infty} \mathbf{b}_i(\mathbf{z}_2, \mathbf{z}_3) z_1^i \right\rangle \equiv 0. \tag{17}$$

On the $(n - m)$-dimensional plane $\mathbf{z}_3 = 0$, equating the constant terms in z_1 yields

$$\left\langle \frac{\partial a_0(\mathbf{z}_2)}{\partial \mathbf{z}_2}, \mathbf{B}\mathbf{z}_2 + \mathbf{b}_0(\mathbf{z}_2) \right\rangle \equiv 0.$$

Let $a_0(\mathbf{z}_2) = c_0 + c_k(\mathbf{z}_2) + O(k+1)$, where c_0 is a constant, $c_k(\mathbf{z}_2)$ is a homogeneous polynomial of degree k in $\mathbf{z}_2$ with $k \geqslant 1$, and $O(k+1)$ denotes the summation of terms in $\mathbf{z}_2$ with degree larger than k. Substituting $a_0(\mathbf{z}_2)$ into the above equality, we get $\langle \partial c_k(\mathbf{z}_2)/\partial \mathbf{z}_2, \mathbf{B}\mathbf{z}_2 \rangle = 0$, because the lowest degree of the terms in $\mathbf{b}_0(\mathbf{z}_2)$ is larger than 1. So, from lemma 4 and working in a similar way to the proof of theorem 1 we can prove that $c_k(\mathbf{z}_2) = 0$. Consequently, $a_0(\mathbf{z}_2)$ is a constant.

By induction we can prove that all $a_i(\mathbf{z}_2)$ are constants for $i = 1, 2, \ldots$. Hence, $H_1(z_1, \mathbf{z}_2) = H_1(z_1)$. It follows from (17) that either $H_1(z_1, \mathbf{z}_2) \equiv 0$ or $h_1(z_1, \mathbf{0}_3) \equiv 0$.

If $h_1(z_1, \mathbf{0}_3) \not\equiv 0$ then $H(z_1, \mathbf{z}_2, \mathbf{0}_3) = H_1(z_1, \mathbf{z}_2) \equiv 0$. We have $H = H_2(z_1, \mathbf{z}_2, \mathbf{z}_3)$. Set $H_2(z_1, \mathbf{z}_2, \mathbf{z}_3) = \sum_{|\mathbf{s}|=\mathbf{1}}^{\infty} g_{\mathbf{s}}(z_1, \mathbf{z}_2)\mathbf{z}_3^{\mathbf{s}}$, where $\mathbf{z}_3^{\mathbf{s}} = z_{n-m+1}^{s_1} \dots z_n^{s_m}$ and $|\mathbf{s}| = s_1 + \dots + s_m$. Since the first integral $H = H_2$ satisfies

$$\frac{\partial H_2}{\partial z_1} h_1(z_1, \mathbf{z}_3) + \left\langle \frac{\partial H_2}{\partial z_2}, \mathbf{B}\mathbf{z}_2 + \mathbf{h}_2(z_1, \mathbf{z}_2, \mathbf{z}_3) \right\rangle \equiv 0$$

h_1 and $\mathbf{h}_2$ have the lowest degrees larger than 1, and $h_1(z_1, \mathbf{0}_3) \not\equiv 0$, by induction and working in similar way to the previous proof we can get that all $g_{\mathbf{s}}(z_1, \mathbf{z}_2) =$ constant. This means that the first integral $H = H_2(z_1, \mathbf{z}_2, \mathbf{z}_3)$ of system (16) is a formal series in $\mathbf{z}_3$ only in a neighbourhood of the singularity $\mathbf{0}$. It is in contradiction with the assumption.

Now, we suppose that $h_1(z_1, \mathbf{z}_2, \mathbf{0}_3) = h_1(z_1, \mathbf{0}_3) \equiv 0$. By selecting a sufficiently higher cut of the formal series transformation in the Poincaré–Dulac theorem, and working in a similar way to the proof of lemma 6 in [14], we can prove that the singularity $\mathbf{x} = \mathbf{0}$ of system (2) on the $(n - m)$-dimensional analytic manifold S is not isolated if and only if $h_1(z_1, \mathbf{z}_2, \mathbf{0}_3) \equiv 0$. So, we have proved the 'only if' part of statement (a).

Using the statement of the last paragraph, we can get also the proof of the 'if' part of statement (a).

Proof of statement (b). From statement (a) it follows easily that the 'only if' part holds. We now prove the 'if' part.

Working in a similar way to the proof of statement (a), we only need to consider system (15). The condition $n - m = 2$ means that the matrix $\mathbf{B} = \mu_2$. Since the singularity $\overline{\mathbf{y}} = \mathbf{0}$ of system (15) on the two-dimensional plane $(\overline{y}_1, \overline{y}_2)$ is not isolated, there exists a sufficiently small neighbourhood U of $\mathbf{0}$ such that the equation $\mu_2\overline{y}_2 + \overline{q}_2(\overline{\mathbf{y}}) = 0$ has a unique solution $\overline{y}_2 = G(\overline{y}_1, \overline{\mathbf{y}}_2)$ and $\overline{q}_1(\overline{y}_1, G(\overline{y}_1, \overline{\mathbf{0}}_2)) \equiv 0$.

Applying the change of variables

$$z_1 = \overline{y}_1 \qquad z_2 = \overline{y}_2 - G(\overline{y}_1, \overline{\mathbf{y}}_2) \qquad \mathbf{z}_3 = \overline{\mathbf{y}}_2 \tag{18}$$

to system (15), we get the following system:

$$\dot{z}_1 = p_1(z_1, z_2, \mathbf{z}_3)z_2 + f_1(z_1, z_2, \mathbf{z}_3) \qquad \dot{z}_2 = p_2(z_1, z_2, \mathbf{z}_3)z_2 + f_2(z_1, z_2, \mathbf{z}_3) \qquad \dot{\mathbf{z}}_3 = 0 \tag{19}$$

with $p_2(0, 0, \mathbf{0}_3) = \mu_2 \neq 0$, $f_i(z_1, z_2, \mathbf{0}_3) = 0$, $i = 1, 2$. Obviously, p_1, p_2, f_1 and f_2 are analytic in a neighbourhood of the origin. Moreover, on the plane $\mathbf{z}_3 = 0$ system (19) and the system

$$\dot{z}_1 = p_1(z_1, z_2, \mathbf{z}_3) \qquad \dot{z}_2 = p_2(z_1, z_2, \mathbf{z}_3) \qquad \dot{\mathbf{z}}_3 = 0 \tag{20}$$

have the same first integrals. Since system (20) is regular at the origin of the plane $\mathbf{z}_3 = 0$, by the box flow theorem it has an analytic first integral on the plane $\mathbf{z}_3 = \mathbf{0}$ in a sufficiently small neighbourhood of the origin. Hence, system (19) has also an analytic first integral on the plane $\mathbf{z}_3 = 0$ in the corresponding neighbourhood of $(0, 0, \mathbf{0}_3)$.

From the assumption of the theorem, i.e. the surface formed by the singularities of system (2) is transverse to S, and passes through the origin, it follows that the singularities of (19) form a surface, denoted by S', transverse to the plane $\mathbf{z}_3 = \mathbf{0}_3$. Hence, there exists a neighbourhood V of $\mathbf{z} = \mathbf{0}$ such that for each $\mathbf{c}_3 \neq \mathbf{0}_3$ with $\|\mathbf{c}_3\|$ sufficiently small, the curve $S' \cap \{\mathbf{z}_3 = \mathbf{c}_3\} \cap V$ on the invariant plane $\{\mathbf{z}_3 = \mathbf{c}_3\}$ is formed by the singularities of system (19). Working in a similar way to the proof of the last paragraph we can prove that system (19) has an analytic first integral in $\{\mathbf{z}_3 = \mathbf{c}_3\} \cap V$. Therefore, by continuity of solutions and Hartog's theorem [11] we obtain that system (19) has an analytic first integral in a neighbourhood of

$\mathbf{z} = \mathbf{0}$, which consists of $\mathbf{z}_3$ and at least one of z_1 and z_2. Since the transformation (18) is an analytic diffeomorphism, system (15) has an analytic first integral of the form

$$H^* = \sum_{|\mathbf{s}|=\mathbf{1}}^{\infty} C_{\mathbf{s}}(\mathbf{y}) \left(\Phi_1^*(\mathbf{My})\right)^{s_1} \ldots (\Phi_m^*(\mathbf{My}))^{s_m}$$

with $C_{\mathbf{s}}$ not all constants. This completes the proof of statement (b). We have finished the proof of the theorem.

5. Proof of theorem 3

Since system (1) is semi-quasi-homogeneous of degree l with exponents $s_1, \ldots, s_n$ having $\mathbf{F}$ of form (8), under the transformation

$$\mathbf{x} \longrightarrow \rho^{\mathbf{S}}\mathbf{x} \qquad t \longrightarrow \rho^{-(l-1)}t \qquad \rho^{\mathbf{S}} = \operatorname{diag}(\rho^{s_1}, \ldots, \rho^{s_n}) \tag{21}$$

system (1) becomes

$$\dot{\mathbf{x}} = \mathbf{F}_l(\mathbf{x}) + \widetilde{\mathbf{F}}(\mathbf{x}, \rho) \tag{22}$$

where $\widetilde{\mathbf{F}}(\mathbf{x}, \rho)$ is a vector-valued formal series with respect to ρ or ρ^{-1} according that the system is positively or negatively semi-quasi-homogeneous. We note that if $H(\mathbf{x})$ is a first integral of system (1), then so is $\rho^k H(\mathbf{x})$ for $k \in \mathbb{Z}$. Using the transformation (21) we can get a first integral $H(\rho^{\mathbf{S}}\mathbf{x})$ of system (22). We assume that the first integral $H(\mathbf{x})$ under the change (21) has the form

$$H(\mathbf{x}, \rho) = H_r(\mathbf{x}) + \rho H_{r+1}(\mathbf{x}) + \rho^2 H_{r+2}(\mathbf{x}) + \cdots$$

or

$$H(\mathbf{x}, \rho) = H_r(\mathbf{x}) + \rho^{-1} H_{r-1}(\mathbf{x}) + \rho^{-2} H_{r-2}(\mathbf{x}) + \cdots$$

depending on system (1) is positively or negatively semi-quasi-homogeneous respectively, where $H_j(\mathbf{x}) = 0$ for $j \leqslant 0$, and $H_j(\rho^{\mathbf{S}}\mathbf{x}) = \rho^j H_j(\mathbf{x})$ with $j > 0$. We call $H_r(\mathbf{x})$ the first term of $H(\mathbf{x}, \rho)$.

If $H(\mathbf{x})$ is a first integral of system (1), then from the expressions of $\mathbf{F}$ and H, and the definition of first integrals we can obtain that $H_r(\mathbf{x})$ is a polynomial first integral of the quasi-homogeneous cut of system (22):

$$\dot{\mathbf{x}} = \mathbf{F}_l(\mathbf{x}). \tag{23}$$

Using the change of variable $\mathbf{x} = t^{-\mathbf{W}}(\overline{\mathbf{c}} + \mathbf{u})$, where $\overline{\mathbf{c}}$ satisfies $\mathbf{F}_l(\overline{\mathbf{c}}) + \mathbf{W}\,\overline{\mathbf{c}} = 0$, we get that

$$H_r(\mathbf{x}) = t^{-r/(l-1)} H_r(\overline{\mathbf{c}} + \mathbf{u}) = \overline{H}_r(u_0, \mathbf{u})$$

where we select $u_0 = t^{-1/(l-1)}$ as a new auxiliary variable. System (23) becomes

$$t\dot{\mathbf{u}} = \mathbf{K}\mathbf{u} + \overline{\mathbf{F}}_l(\mathbf{u}) \qquad \overline{\mathbf{F}}_l(\mathbf{u}) = \mathbf{W}\overline{\mathbf{c}} + \mathbf{F}_l(\overline{\mathbf{c}} + \mathbf{u}) - \frac{\partial \mathbf{F}_l(\overline{\mathbf{c}})}{\partial \mathbf{x}}\mathbf{u}$$

where $\mathbf{K}$ is the Kovalevskaya matrix associated with the balance $\overline{\mathbf{c}}$.

Let $\tau = \log t$. Then $\overline{H}_r(u_0, \mathbf{u})$ is a polynomial first integral of the system

$$u_0' = -\frac{1}{l-1}u_0 \qquad \mathbf{u}' = \mathbf{K}\mathbf{u} + \overline{\mathbf{F}}_l(\mathbf{u}) \tag{24}$$

where the prime denotes the derivative with respect to τ.

On the other hand, we can prove easily that the eigenvalues of the linear part of system (24) at $u_0 = 0$ and $\mathbf{u} = 0$ are $\lambda_0 = -1/(l-1), \lambda_1 = 0, \lambda_2, \ldots, \lambda_n$. Since -1 is always a Kowalevskaya exponent, let $\lambda_2 = -1$. From the assumptions of the theorem we obtain that the linear algebraic equation in $\lambda_2, \ldots, \lambda_n$

$$-k_0 + (l-1)\sum_{i=2}^{n} k_i\lambda_i = (k_0 + (l-1)k_2)\lambda_2 + (l-1)\sum_{i=3}^{n} k_i\lambda_i = 0$$

with $k_i \in \mathbb{Z}^+, k_0 + (l-1)\sum_{i=2}^{n} k_i \neq 0$, has m-independent solutions. By the assumption, equation $\mathbf{K}\mathbf{u} + \overline{\mathbf{F}}_l(\mathbf{u}) = \mathbf{F}_l(\overline{\mathbf{c}} + \mathbf{u}) + \mathbf{W}(\overline{\mathbf{c}} + \mathbf{u}) = \mathbf{0}$ has the isolated root $\mathbf{u} = \mathbf{0}$, so the origin as the singularity of system (24) is isolated. Hence, it follows from theorem 2 that the first integral $\overline{H}_r(u_0, \mathbf{u})$ of system (24) in a neighbourhood of the origin depends only on $\Phi_1, \ldots, \Phi_m$.

It is easy to know that any smooth function in $\Phi_1, \ldots, \Phi_m$ is also a first integral of (22). If $H(\mathbf{x}, \rho)$ is a first integral of system (22), and it depends not only on $\Phi_1, \ldots, \Phi_m$, we can select a first integral $\Phi(\mathbf{x}, \rho)$ containing only $\Phi_1, \ldots, \Phi_m$ of (22) such that the first quasi-homogeneous term $\mathcal{H}$ of $H(\mathbf{x}, \rho) - \Phi(\mathbf{x}, \rho)$ contains components depending not only on Φ_i, $i = 1, \ldots, m$. This is in contradiction with the argument of the last paragraph. This proves the theorem.

6. Conclusion

In this section we summarize our main results and methods. In theorem 1 we generalize theorem A given in [13] to the case that the matrix of the linear part of the system at a singularity can be non-diagonalizable. The main method in our proof is to use the spectrum of a linear operator on the linear space formed by the homogeneous polynomials of the same degree.

Our second theorem extends the results given in theorem 1 of [14]. We give a necessary and sufficient condition for a system with some analytic first integrals to have other first integrals independent of the given first integrals. In the proof we combine the spectrum of linear operators and the Poincaré–Dulac normal form.

Our theorem 3 provides an extension of theorem B given in [13]. In our results the Kowalevskaya exponents independent of the given first integrals can be zero. The proof is to combine the singular analysis and the methods given in the proof of theorems 1 and 2.

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